Math 299 — **Lecture 24: Sections 9.4 -9.5**
Bijective Functions and Composition of Functions

Bijective functions

Definition: A function $f : X \to Y$ is **bijective** (or **one–to–one correspondence**)

if f is both injective and surjective.

Theorem. If A and B are finite sets with $|A| = |B| = n$, then there are $n!$ bijective functions from A to B.

Theorem. Let A and B be finite nonempty sets with $|A| = |B|$ and let f be a function from A to B. Then f is one-to-one if and only if f is onto.

- Does this hold if A and B are infinite sets?

Examples:

1. Prove that the function $f : \mathbb{R} - \{5\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x}{x-5}$ is bijective.

2. Prove that the function $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ defined by $f([x]) = [5x + 2]$ is a well defined bijective function.

Composition of Functions

Definition: If $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f$ is a function from A to C defined by $(g \circ f)(x) = g(f(x))$. It is called the **composition** of f and g.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x + 5$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Find the domain and range of f and g, as well as $f \circ g$ and $g \circ f$ (where they are defined).

2. $f = \{(1, m), (2, n), (3, m)\}$, $g = \{(k, 1), (l, 2), (m, 1), (n, 3)\}$. Find $f \circ g$ and $g \circ f$.

Theorem. Let $f : A \to B$ and $g : B \to C$ be two functions.

(a) If f and g are injective, then so is $g \circ f$.

(b) If f and g are surjective, then so is $g \circ f$.

Corollary. Let $f : A \to B$ and $g : B \to C$ be bijective functions, then so is $g \circ f$ is bijective.